12/8

Recall: If R is "nice" and ∂R is also nice and is a v.f. on $\mathbb{R}^3$ w/ cts partials, then
$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$$

Ex. Calculate flux of $\vec{F} = \langle 3x, xy, 2xz \rangle$ across the surface of the cube $R = [0,1]^3$

Sol: We apply the divergence theorem

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$$
$$= \iiint_R (3 + x + 2x)\, dV$$
$$= \iiint_R (3+3x)\, dV$$
$$= \int_0^1 \int_0^1 \int_0^1 (3+3x)\, dx\, dy\, dz$$
$$= \int_0^1 \int_0^1 \left[3x + \tfrac{3}{2}x^2\right]_0^1 dy\, dz$$
$$= \int_0^1 \int_0^1 \tfrac{9}{2}\, dy\, dz$$
$$= \int_0^1 \left[\tfrac{9}{2}y\right]_0^1 dz$$
$$= \int_0^1 \tfrac{9}{2}\, dz = \left[\tfrac{9}{2}z\right]_0^1 = \tfrac{9}{2} \quad \boxed{\checkmark}$$

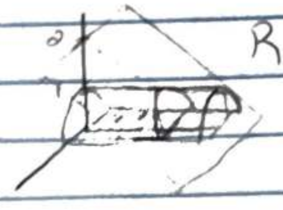
Ex. Calculate flux of $\vec{F} = \langle x^2yz, xy^2z, xyz^2 \rangle$ across the boundary of the rectangular box $R = [0,a] \times [0,b] \times [0,c]$ for constants $a, b, c > 0$

Sol: We apply the divergence theorem

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$$
$$= \iiint_R (2xyz + 2xyz + 2xyz)\, dV$$
$$= 6 \iiint_R xyz\, dV$$
$$= 6 \int_0^c \int_0^b \int_0^a xyz\, dx\, dy\, dz$$
$$= 6 \left(\int_0^c z\, dz\right)\left(\int_0^b y\, dy\right)\left(\int_0^a x\, dx\right)$$
$$= \tfrac{6}{8}\left[z^2\right]_0^c \left[y^2\right]_0^b \left[x^2\right]_0^a = \tfrac{6}{8}(a^2-0)(b^2-0)(c^2-0) = \tfrac{3}{4}a^2b^2c^2 \quad \boxed{\checkmark}$$

Ex. Calculate $\iint_S \vec{F}\cdot d\vec{S}$ of $\vec{F}=\langle xy, y^2+e^{xz^2}, \sin xy\rangle$ across the surface bounding region w/ $z=1-x^2$, $z=0$, $y=0$, $y+z=2$

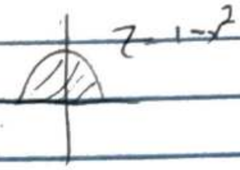
Sol: Again use divergence theorem



$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial R} \vec{F}\cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\,d\vec{S}$$

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial y}[y^2+e^{xz^2}] + \frac{\partial}{\partial z}[\sin xy]$$
$$= y + 2y + 0 = 3y$$

Parameterize R: shadow in xz-plane



$R=\{(x,y,z): -1\le x\le 1,\ 0\le z\le 1-x^2,\ 0\le y\le 2-z$

$1-x^2=0$ iff $x=\pm 1$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \iiint_R 3y\,dV = \int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} 3y\,dy\,dz\,dx$$
$$= \int_{-1}^{1}\int_0^{1-x^2}\left[\tfrac{3}{2}y^2\right]_0^{2-z} dz\,dx$$
$$= \tfrac{3}{2}\int_{-1}^{1}\int_0^{1-x^2}(2-z)^2\,dz\,dx$$
$$= -\tfrac{1}{2}\int_{-1}^{1}\left[(2-z)^3\right]_0^{1-x^2} dx$$
$$= -\tfrac{1}{2}\int_{-1}^{1}\left[(2-(1-x^2))^3-(2-0)^3\right]dx$$
$$= -\tfrac{1}{2}\int_{-1}^{1}\left[(1+x^2)^3-8\right]dx$$
$$= -\tfrac{1}{2}\int_{-1}^{1}(1+3x^2+3x^4+x^6-8)\,dx$$
$$= -\tfrac{1}{2}\left[x^3+\tfrac{3}{5}x^5+\tfrac{1}{7}x^7-7x\right]_{-1}^{1}$$
$$= -\tfrac{2}{2}\left(1+\tfrac{3}{5}+\tfrac{1}{7}-7\right) \quad \square$$

Ex. Compute flux of $\vec{F}=\langle xye^z, xy^2z^3, -ye^z\rangle$ across S surface of box bdd by coordinate planes and $x=3$, $y=2$, $z=1$

Sol: Apply divergence theorem, noting $S=\partial R$ for $R=[0,3]\times[0,2]\times[0,1]$ and $\operatorname{div}(\vec{F}) = ye^z + 2xyz^3 - ye^z = 2xyz^3$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \iiint_R 2xyz^3 = 2\left(\int_0^3 x\,dx\right)\left(\int_0^2 y\,dy\right)\left(\int_0^1 z^3\,dz\right)$$
$$= 2\left[\tfrac{1}{2}x^2\right]_0^3\left[\tfrac{1}{2}y^2\right]_0^2\left[\tfrac{1}{4}z^4\right]_0^1$$
$$= \tfrac{1}{8}(3^2-0)(2^2-0)(1^2-0) = \tfrac{9}{2} \quad \square$$

Ex. Compute flux of $\vec{F}=\langle z, y, zx\rangle$ across surface enclosed by coordinate planes and plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ for constant $a,b,c>0$

$\vec{n}\cdot(\vec{x}-\vec{p})=0$

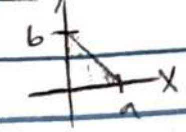
$\vec{n}\cdot\vec{x}=d$

$\langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c}\rangle\cdot\langle x,y,z\rangle = 1$

Parameterize the tetrahedron:

$R = \{(x,y,z) \mid 0 \le x \le a,\ 0 \le y \le b(1-\frac{x}{a}),$

$0 \le z \le c(1-\frac{x}{a}-\frac{y}{b})\}$



$\frac{x}{a}+\frac{y}{b}=1$

and $div(\vec F) = 0+1+x = 1+x$

$\therefore$ by divergence theorem:

$\iint_S \vec F\cdot d\vec S = \iint_{\partial R} \vec F\cdot d\vec S = \iiint_R div(\vec F)\, dV$

$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} (1+x)\, dz\, dy\, dx$

$= \int_0^a (1+x) \int_0^{b(1-\frac{x}{a})} [z]_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} = \int_0^a (1+x) \int_0^{b(1-\frac{x}{a})} c(1-\frac{x}{a}-\frac{y}{b})\, dy\, dx$

$= c\int_0^a (1+x)\left[y - \frac{xy}{a} - \frac{1}{2b}y^2\right]_0^{b(1-\frac{x}{a})} dx$

$= c\int_0^a (1+x)\left[b(1-\frac{x}{a})(1-\frac{x}{a}-\frac{1}{2b}b(1-\frac{x}{a}))\right] dx$

$= bc\int_0^a (1+x)(1-\frac{x}{a})((1-\frac{x}{a})-\frac{1}{2}(1-\frac{x}{a}))\, dx$

$= \frac{1}{2}bc\int_0^a (1+x)(1-\frac{x}{a})^2\, dx$

$= \frac{1}{2}bc\int_0^a (1+(1-\frac{1}{a}-\frac{1}{a})x + (-\frac{1}{a}-\frac{1}{a}+\frac{1}{a^2})x^2 + \frac{1}{a^2}x^3\, dx$

$= \frac{1}{2}bc\int_0^a 1+(1-\frac{2}{a})x + (\frac{1}{a^2}-\frac{2}{a})x^2 + \frac{1}{a^2}x^3)\, dx$

$= \frac{1}{2}bc\left[x + \frac{1}{2}(1-\frac{2}{a})x^2 + \frac{1}{3}(\frac{1}{a^2}-\frac{2}{a})x^3 + \frac{1}{4a^2}x^4\right]_0^a$

$= \frac{1}{2}bc\left[a + \frac{a^2}{2}(1-\frac{2}{a}) + \frac{a^3}{3}(\frac{1}{a^2}-\frac{2}{a}) + (\frac{1}{4}a^2) - 0\right]$

$= \frac{1}{2}abc\left(1 + \frac{1}{2}(a-2) + \frac{1}{3}(1-2a) + \frac{1}{4}a\right)$

$= \frac{1}{2}abc\left(\frac{1}{3} + a(\frac{1}{2}-\frac{2}{3}+\frac{1}{4})\right)$ ∎

Ex. Compute the flux of $\vec F = \langle 2x^3+y^2, y^2+z^2, 3y^2 z\rangle$ across the surface of the region bdd by paraboloid $z = 1-x^2-y^2$ and plane $z=-3$

Sol: Apply the divergence theorem

$div(\vec F) = 6x^2+3y^2+3y^2$

$= 6(x^2+y^2)$

$z=-3$

Parameterize R: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$-3 = 1-x^2-y^2$

$x^2+y^2=4$

$R_{cyl} = \{(r,\theta,z) \mid 0 \le r \le 2$

$0 \le \theta \le 2\pi$

$-3 \le z \le 1-r^2$

$\therefore \iint_{\partial R} \vec F\cdot d\vec S = \iiint_R div(\vec F)\, dV$

$= \iiint_{R_{cyl}} div(\vec F)_{(cyl)}\, r\, dV_{cyl}$

$$= \int_0^{2\pi} \int_0^2 \int^{1-r^2} 6r^2 \, r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^2 6r^3 \left[z\right]_{-3}^{1-r^2} dr$$

$$= 12\pi \int_0^2 r^3 \left[1 - r^2 - (-3)\right] dr \qquad r^2 = 4-u \qquad u(2) = 0, \; u(0) = 4$$

$$= 12\pi \int_0^2 r \cdot r^2 (4 - r^2) \, dr \qquad u = 4 - r^2, \; du = -2r\,dr$$

$$= -6\pi \int_4^0 (4-u)u \, du = 6\pi \int_0^4 4u - u^2 \, du$$

$$= 6\pi \left(2u^2 - \frac{1}{3}u^3\right)\Big|_0^4$$

$$= 6\pi \left(32 - \frac{64}{3}\right) \; \boxed{}$$